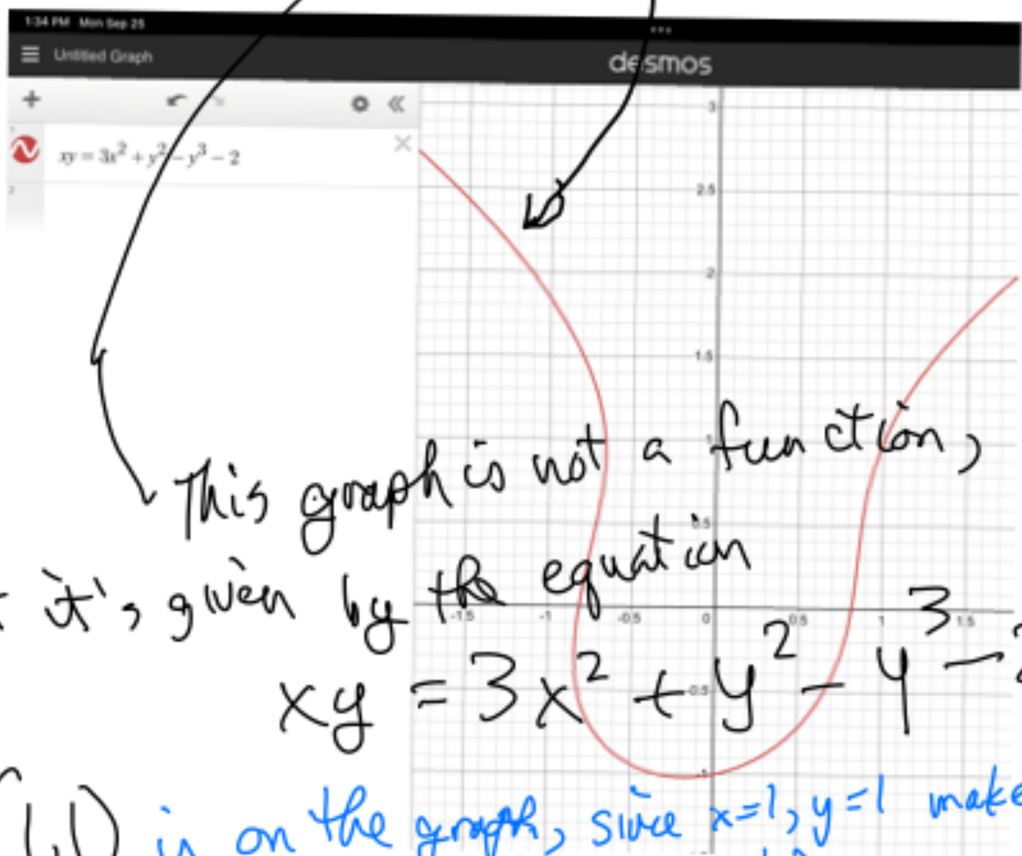
Last time:



This graph is not a function, but it's given by the equation

$$xy = 3x^2 + y^2 - y^3 - 2.$$

$(1,1)$ is on the graph, since $x=1, y=1$ makes the equation true.

Questions: ① How would we find $\frac{dy}{dx}$ (ie slope of tangent line to the graph?)

② In particular, what is the equation of the tangent line to the graph at $(1,1)$?

③ At what points does this graph have a vertical tangent line?

TRICK:
- Pretend y is a function of x
- Compute the derivative of both sides of the equation
- Use chain rule when necessary
- Solve for y'.

Solution: $xy = 3x^2 + y^2 - y^3 - 2$.

Derivative:

$$(x)'y + x(y)' = 6x + (y^2)' - (y^3)' - 0$$

$$\Rightarrow \quad y + xy' = 6x + 2y \cdot y' - 3y^2 \cdot y'$$

$$\Rightarrow \quad y - 6x = -xy' + 2yy' - 3y^2y'$$

$$\Rightarrow \quad y - 6x = y'(-x + 2y - 3y^2)$$

$$\Rightarrow \quad \text{①} \quad \boxed{\frac{y - 6x}{-x + 2y - 3y^2} = y'} \qquad \frac{dy}{dx} = y'$$

② Eqn of tangent line @ $(x,y) = (1,1)$.

$$\Rightarrow \quad y' = \frac{1 - 6\cdot 1}{-1 + 2\cdot 1 - 3\cdot 1^2} = \frac{-5}{-2} = \boxed{\frac{5}{2}}$$

$$(y-y_0)=m(x-x_0)$$

$$y-1=\frac{5}{2}(x-1) \Rightarrow y=\frac{5}{2}x-\frac{5}{2}+1$$

$$\boxed{y=\frac{5}{2}x-\frac{3}{2}}$$

(3) When does this graph have a vertical tangent line?

denom $=0$ $\quad -x+2y-3y^2=0$

$$x=2y-3y^2$$

orig equation $\quad xy=3x^2+y^2-y^3-2.$

$$(2y-3y^2)y=3(2y-3y^2)^2+y^2-y^3-2$$

in theory, we would solve this for y.

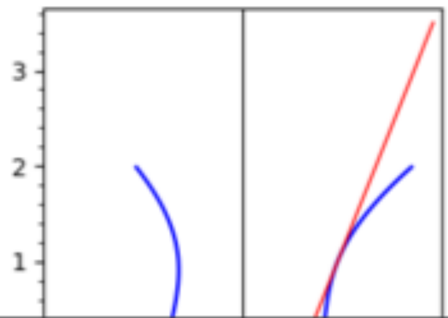
<u>sagemath:</u> . . .

Type some Sage code below and press Evaluate.

```
var('x,y') # this tells sagemath that x and y are
# symbolic variables
eq = x*y == 3*x^2+y^2-y^3-2
a = implicit_plot(eq,(-2,2),(-2,2),axes=True)
b = plot(5/2*x-3/2,(x,-1,2),color="red")
show(eq)
show(a+b)
# look for vertical tangent lines
# where denom of y' = 0
# x = 2*y-3*y^2
```

Evaluate

$$xy = -y^3 + 3x^2 + y^2 - 2$$




```
# x = 2*y-3*y^2
eq2 = eq.subs(x = 2*y-3*y^2)
show(eq2)
# let's solve eq2
bubba = solve(eq2,y)
show(bubba)
answ = [ oink.rhs().n() for oink in bubba ] # called list
              # comprehension in python

show(answ)
```

extracts right hand side of each equation & makes it numeric

```
show(answ)
y1 = answ[0] # first element of answ
y2 = answ[1] # second element of answ
x1 = 2*y1 - 3*y1^2
x2 = 2*y2 - 3*y2^2
plot(a+point((x1,y1),color="pink",size=50)+point((x2,y2),color="green",size=50))
```

points where tangent line is vertical.

$$-(3\,y^2 - 2\,y)y = -y^3 + 3\,(3\,y^2 - 2\,y)^2 + y^2 - 2$$

$$\left[y = -\frac{1}{54}\sqrt{\frac{9\left(12\sqrt{505931}\sqrt{3} - 8497\right)^{\frac{2}{3}} + 91\left(12\sqrt{505931}\sqrt{3} - 8497\right)^{\frac{1}{3}} - 4743}{\left(12\sqrt{505931}\sqrt{3} - 8497\right)^{\frac{1}{3}}}} - \frac{1}{2}\sqrt{-\frac{1}{81}\left(12\sqrt{505931}\sqrt{3} - 8497\right)^{\frac{1}{3}} + \frac{5\cdots}{81\left(12\sqrt{50593\cdots}\right.}}\right.$$

$$[-0.293080446416919, 0.909827206368958, 0.321256249653610 - 0.417835961944831i, 0.321256249653610 + 0.417835961944831i]$$